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Modelling Policy Change in Endogenous Growth Theory

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ABSTRACT

Innovation-based endogenous growth (IBEG) models are often used for evaluating R&D related policies. In such evaluations different policies correspond to different long-run balanced growth paths. If the model contains predetermined state variables, whose values cannot change discontinuously, the model economy cannot immediately move to a new long-run equilibrium after policy change. However, the transition paths from one long-run equilibrium to another have until now been left unanalyzed in the context of many important IBEG models. The theory of dynamic systems provides tools for such analyses. We review theory of dynamical systems and the development of IBEG models. We apply the theory of dynamical systems to Romer's model of growth by specialization, and we recapitulate the reasons that make the transition paths of the first quality ladder models trivial and unique. We point out that the transition paths of many newer quality ladder models pose much more challenging problems.

JEL Codes: E10, O32, O38, O41

Keywords:
endogenous growth theory, R&D policy, dynamical systems, economic growth, productivity

TIIVISTELMÄ

Politiikkamuutosten mallinnus endogeenisessa kasvuteoriassa

Innovaatioperustaisia endogeenisia kasvumalleja (IBEG-malleja) käytetään usein T&K-toimintaan liittyvien politiikkatoimien arvioimiseen. Tällöin eri politiikkavaihtoehdoilla on vastineenaan toisistaan poikkeavia tasaisen kasvun uria. Jos mallissa on tilamuuttujia, joiden arvot eivät voi hyppäyksenomaisesti muuttua, mallin kuvaama talous ei voi siirtyä uuteen pitkän tähtäimen tasapainoon heti politiikan muututtua. On kuitenkin monia tärkeitä IBEG-malleja, joiden tasaisen kasvun urien välisiä siirtymäpolkuja ei ole analysoitu yksityiskohtaisesti. Dynaamisten systeemien teoria tarjoaa työkaluja siirtymäpolkujen analysointiin. Esittelemme dynaamisten systeemien teoriaa ja IBEG-mallien kehitystä. Sovellamme dynaamisten systeemien teoriaa Romerin erikoistumisen kautta tapahtuvan kasvun malliin, ja selitämme miksi ensimmäisten luovan tuhon kautta tapahtuvaa kasvua kuvaavien quality ladder -mallien siirtymäpolut olivat yksikäsitteisiä ja triviaaleja. Huomautamme, että monien uudempien luovan tuhon mallien siirtymäpolkujen ratkaiseminen olisi paljon haastavampaa.

JEL-koodit: E10, O32, O38, O41

Avainsanat:
endogeeninen kasvuteoria, T&K-politiikka, dynaamiset systeemit, talouskasvu, tuottavuus

1. Introduction

Innovation-based endogenous growth (IBEG) models aim at explaining productivity growth as a result of research and development. They have often been criticized because of their weak correspondence with reality. For example, it is difficult to explain the observed GDP growth or total factor productivity growth with the amount of R&D in the United States, since growth has stayed constant or even decreased in the US while the amount of R&D has dramatically increased (see e.g. Bloom et al., 2020; Jones, 1995; Segerstrom, 1998). A criticism which is, perhaps, somewhat less common, is concerned with the dynamics of IBEG models.

In many policy applications of IBEG models one has rested content with comparing the *balanced growth paths* that correspond to the contrasted policies. In other words, the effects of different policies (such as e.g. different R&D subsidies) have been compared by comparing the GDP growth rates and other relevant features of the economy that correspond to them in the long run. This is, however, problematic when the economy of the model cannot immediately "jump" from one balanced growth path to another one.

Some of the variables of a growth model are usually predetermined *state variables* such as capital, which cannot change their values discontinuously, while others are *control variables* such as private consumption, for which such changes are possible. If the evolution of a state variable after a policy change is left unanalyzed, it remains unclear how fast the system would transition to its new long-run equilibrium after the policy change. It might even be unclear whether the assumptions of the model suffice to fix a unique transition path to the long-run equilibrium, or whether it might allow for an infinite variety of transition paths.

(Atkeson & Burstein, 2019) have formulated a growth model which nests several important IBEG models as its special cases. They use their model for analyzing the growth effects of increasing R&D personnel over a 20-year horizon, and they deduce the elasticities of productivity and aggregate output relative to R&D in a variety of model specifications. In the chosen specifications a policy change which leads to a permanent increase of R&D personnel by 10 per cent would cause within the next 20 years only an increase between 1.6 and 5.2 percent to the total factor productivity, and only an increase between 0.3 and 4.2 percent to aggregate production. These values are much smaller than the corresponding values on relevantbalanced growth paths.[1] Hence, even if some policy measure should increase total factor productivity or production in the long run, they do not necessarily have any remarkable growth effects in the time perspective within which various policy measures are normally compared.

The determination of transition paths poses problems also for many other macroeconomic models. It is a standard practice to determine the transition paths of the Ramsey-Cass-Koopmans neoclassical growth model using optimal control theory. Detailed discussions of the analogous problems are readily available also for overlapping generations models and for DSGE models (see e.g. Auerbach & Kotlikoff, 1987; Fernández-Villaverde et al., 2016).

DSGE models are, by definition, stochastic, and the economies described by them do not stay in their steady states even if they were in such a state originally. To solve a DSGE model, one must specify how its control variables react to the exogenous stochastic shocks. It is generally known that the solution to this problem does not, in general, have to be unique. The solution might be non-existent, in which the economy of the model is instable, and there may also be an infinite variety of such solutions. In the latter case one cannot determine a unique equilibrium for the DSGE model in question (Fernández-Villaverde et al., 2016, p. 553) and it has sunspot equilibria (see e.g. Lubik & Schorfheide, 2003).

Despite of their obvious importance the transition paths of IBEG models have not been discussed as systemically as the transition paths of the neoclassical growth model and its variants, and of various DSGE models. In what follows I shall first argue that the *theory of dynamical systems* (rather than optimal control theory) would be a natural framework for analyzing transition paths of IBEG models. After illustrating the theory of dynamical systems by solving the neoclassical growth model's transition path using it, I shall review the most important types of endogenous growth models and consider the question whether their transition paths are trivial, non-trivial but known, or (to the best of our knowledge) still unanalyzed.

[1] Atkeson – Burstein (2019), p. 2661. It should be observed that these results are conditional: they state how much TFP and output would grow in the considered model if research labour was increased by a fixed percentage. They do not, however, answer the question how much according to the considered model various reforms would change the amount of research labour. In other words, the results are conditional on the assumption that a transition path of a specific kind exists, but they do not prove its existence or uniqueness.

Similarly with most other macroeconomic models, the endogenous growth models which are considered below can be formulated in either continuous or discrete time. To unify notation, I shall use the continuous-time version for all of them.

2. Formulating the problem

We consider a system which can be characterized with a finite number of real-valued variables at each instant of time. We assume that the variables can be divided into predetermined *state variables* whose values are at each instant of time determined by the earlier development of the economy, and *control variables*, which may be subject to abrupt changes e.g. do to a policy change. More rigorously, we shall consider a system which can at each moment of time $t \geq 0$ be represented by the vector of state variables $\mathbf{x}(t) = \left(x_1(t),...,x_{n_x}(t)\right)^T$ and the vector of control variables $\mathbf{y}(t) = \left(y_1(t),...,y_{n_y}(t)\right)^T$. We define a vector which contains both types of variables as

(1) $$\mathbf{z}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix}$$

In addition, we allow for the case in which all the variables are control variables. In this case $\mathbf{z} = \mathbf{y}$ and $n_x = 0$.

A *balanced growth path* of a macroeconomic model corresponds to a situation in which suitably chosen normalized variables stay constant, and in the economic application of the above framework the variables in (1) should be taken to be such normalized variables. We shall refer to the balanced growth path also as a *steady state* of the normalized variables. Normally, the equilibrium conditions of a deterministic macroeconomic model suffice to determine the time derivative $\dot{\mathbf{z}}$ for any given combination $\mathbf{z} = \left[\mathbf{x}(t) \;\; \mathbf{y}(t)\right]^T$. This implies that one may view the derivative

(2) $$\dot{\mathbf{z}} = Z(\mathbf{z})$$

as a function Z of **z**. When this function and the value $\mathbf{z}(t_0)$ has been given, the values $\mathbf{z}(t)$ will be determined for all $t \geq t_0$. Clearly, the values $\mathbf{z}^*$ correspond to a balanced growth path if

(3) $$Z(\mathbf{z}^*) = \mathbf{0}$$

In general, the initial values $\mathbf{z}_0$ correspond to a *transition path* of the model if the assumptions $\mathbf{z}(t_0) = \mathbf{z}_{t_0}$ and (2) imply that

(4) $$\lim_{t\to\infty} \mathbf{z}(t) = \mathbf{z}^*$$

When the state variables $\mathbf{x}(t_0) = \mathbf{x}_{t_0}$ have been given at some time $t = t_0$, the system has a unique transition path if there is precisely one combination of values $\mathbf{y}(t_0) = \mathbf{y}_{t_0}$ of the control variables for which (1) and (2) imply (4).

This general representation can be illustrated by the neoclassical growth model, which is due to Cass, Koopmans and Ramsey. To choose the simplest possible illustration, we shall consider a continuous-time version of the model with a constant population normalized to 1 and with no productivity growth. In the absence of permanent growth, a normalization of variables is not necessary and the balanced growth path is simply a steady state of the original (unnormalized) variables. We assume that output is given by a time-independent function $f(k)$ where *k* is the available capital, and that the consumers have a logarithmic instantaneous utility function $u(c) = \ln c$ and the utility function

(5) $$C = \int_0^\infty e^{-\rho t} u(c(t))\,dt = \int_0^\infty e^{-\rho t} (\ln c)\,dt$$

In this case the vector **x** of state variables consists of capital *k*, the vector **y** of control variables consists of the aggregate consumption $\tilde{c}$. We consider only symmetric equilibria in which $\tilde{c} = c$. As it is shown in any introduction to growth theory (see e.g. Barro & Sala-i-Martin, 2004) , in this case the function Z of equation (2) is given by

(6) $$Z\left(\begin{bmatrix} k \\ \tilde{c} \end{bmatrix}\right) = \begin{bmatrix} f(k) - \delta k - \tilde{c} \\ \left[f'(k) - \delta - \rho\right]\tilde{c} \end{bmatrix}$$

where δ is the rate of depreciation.

The steady state corresponds to the values $\mathbf{z}^* = \left[k^* \;\; \tilde{c}^*\right]^T$ for which (3) is valid, i.e. for which $Z\left(\left[k^* \;\; \tilde{c}^*\right]^T\right) = 0$. This is the case for the values

(7) $$\begin{cases} f'(k^*) = \delta - \rho \\ \tilde{c}^* = f(k^*) - \delta k^* \end{cases}$$

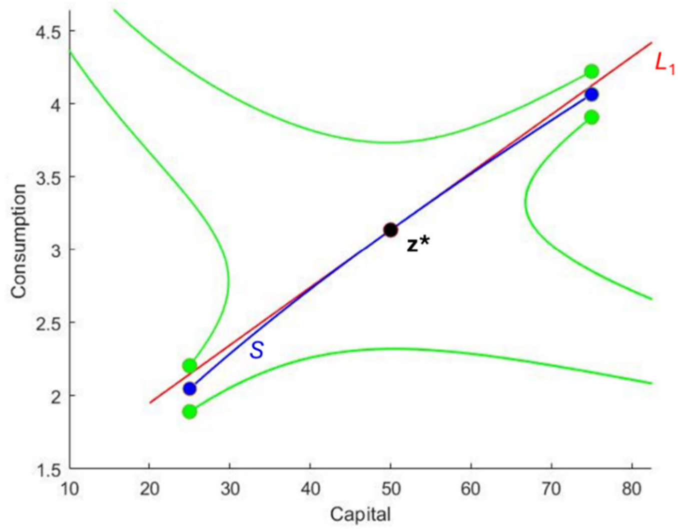
In the context of the neoclassical growth model the existence and uniqueness of the transition path that leads to these values is often demonstrated with a mere graphical representation, such as Figure 1 (a). When a rigorous proof is needed, the analysis is often based on the tools of *optimal control theory*.

When the amount of capital $k(0)$ at $t=0$ has been given, optimal control theory provides a solution to the maximization problem

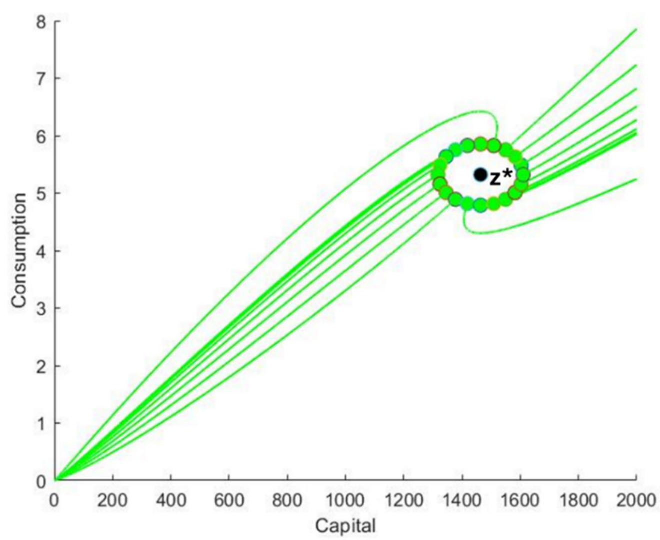
(8) $$C = \max_{c(t),t\geq 0} \int_0^\infty e^{-\rho t} u(c(t))\, dt$$

under the constraints that $\tilde{c} = c$, that (6) is valid, and that the state of the system approaches the fixed point $\left[k^* \;\; \mathrm{c}^*\right]^T$ as $t \to \infty$. However, the solution of this maximization problem does not suffice to demonstrate the uniqueness of the equilibrium thus found. In other words, it does not demonstrate that there could not be other values of *c* for which the system approaches the steady state whenever (6) is valid and at $t(0)$ all consumers chose the consumption $c(0) = c$. If such alternative values of $c(0)$ existed, they would correspond to “bad equilibria” in the sense that each agent could receive a larger utility if all agents changed their strategies, but it would not be optimal for anyone to change her strategy if no one else does so. To prove the uniqueness of transition path, one must appeal to the fact that $\mathbf{z}^*$ is a saddle point, implying that the set (denoted by *S* in Figure 1(a)) of the points from which the system approaches the steady state is one-dimensional (cf. Afonso & Vasconcelos, 2016).

In general, it is not necessarily the case that the set of the points **z** for which (4) is valid has the right dimensionality to guarantee the uniqueness of the transition path. It is not even obvious that a transition path always exists. To illustrate the latter possibility, one may consider a hypothetical case in which capital has increasing returns, i.e. in which $f''(k) > 0$. In this case the fixed point $\mathbf{z}^* = [k^* \; c^*]^T$ would represent a bad equilibrium, since one could achieve a greater welfare by first saving so much that the capital would increase without bounds later on. Figure 1(b) shows the transition paths of a model of this kind. In this case the system cannot move to the steady state if it is not in it originally, and the set for which (4) is valid has just one element, $\mathbf{z}^*$.



(a)



(b)

Figure 1. Paths corresponding to different values of initial capital k_0 and initial consumption c_0 in a simplified neoclassical growth model. Here $\delta = 0.01$ and $\rho = 0.014$, and the production function is of the form $f(k) = Ak^{\alpha}$. Figure 1(a) corresponds to a standard production function with diminishing returns to capital ($A = 1$ and $\alpha = 0.33$), and Figure 1(b) to a production function with increasing returns ($A = 0.05$ and $\alpha = 1.1$).

3. Transition paths and the Invariant Manifold Theorem

We now move to a discussion of the existence and in the uniqueness of transition paths which lead to steady state using the tools of the theory of dynamical systems. For continuous-time models, the most central concept of this theory is the *flow* on a *manifold*.[2] In what follows we will be interested in the manifold $M=\mathbb{R}^{n_x+n_y}$ of all combinations of the state and control variables, as well as in its submanifolds.

When *M* is a differentiable manifold, a flow on *M* is by definition a function $\phi:\mathbb{R}\times M\to M$ which satisfies the conditions $\phi(0,\mathbf{z})=\mathbf{z}$ and $\phi(t+s,\mathbf{z})=\phi(t,\phi(s,\mathbf{z}))$ for any $t,s\in\mathbb{R}$ and $\mathbf{z}\in M$. Intuitively, the value $\phi(t,z)$ is the point to which the system proceeds in the time interval *t* if it originally at the point **z** (Arrowsmith & Place, 1990, p. 12). In what follows, we shall also make use of the notation $\phi_t(\mathbf{z})=\phi(t,\mathbf{z})$.

The function *Z*, which is defined as

(9) $$Z(\mathbf{z})=\left(\frac{d\phi_t(\mathbf{z})}{dt}\right)_{t=0}=\lim_{\Delta t\to 0}\frac{\phi_{\Delta t}(\mathbf{z})-\phi_0(\mathbf{z})}{\Delta t}$$

is called the *field* of the flow ϕ. It is clear that the field *Z* suffices to determine the flow ϕ_t and vice versa. Hence, also each function Z defined by (2), such as e.g. the function (6) which characterizes the neoclassical growth model, is the field of suitably chosen flow $\phi_t(\mathbf{z})$. In this case the flow specifies the values of the capital and consumption after time *t* for each pair of their original values $\mathbf{z}=(k,\tilde{c})$.

The point $\mathbf{z}^*$ is a *fixed point* of the flow ϕ if $\phi_t(\mathbf{z}^*)=\mathbf{z}^*$ for all *t*, and in this case $Z(\mathbf{z}^*)=0$. When some neighborhood U of has been given, the *stable set* W_S of the flow ϕ consists of points *z* for which

$$\lim_{t\to\infty}\phi_t(\mathbf{z})=\mathbf{z}^*$$

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[2] In the case of discrete time model, the flow would be replaced by an iteration of a diffeomorphism of the manifold *M* (Arrowsmith & Place, 1990, pp. 1-2).

Clearly, a unique transition path towards the fixed point exists for some state variable vector **x** if there is a unique control variable vector **y** for which $\mathbf{z}=[\mathbf{x}\ \ \mathbf{y}]^T$ belongs to the stable set. The existence and uniqueness of such values can be investigated by approximating the field *Z* around the fixed point by

(10) $$Z(\mathbf{z}) \approx \Gamma(\mathbf{z}-\mathbf{z}^*)$$

where Γ is the Jacobian

(11) $$\Gamma = \begin{bmatrix} \partial Z_1/\partial z_1 & \dots & \partial Z_1/\partial z_n \\ \vdots & \ddots & \vdots \\ \partial Z_n/\partial z_1 & \dots & \partial Z_n/\partial z_n \end{bmatrix}_{\mathbf{z}=\mathbf{z}^*}$$

It is easy to see that also the flow

(12) $$\bar{\phi}_t(\mathbf{z}) = \mathbf{z}^* + \exp(\Gamma t)(\mathbf{z}-\mathbf{z}^*)$$

has the fixed point $\mathbf{z}^*$ and that for this flow, the derivative defined by (9) satisfies the condition

(13) $$\bar{Z}(\mathbf{z}) = \left(\frac{d\bar{\phi}_t(\mathbf{z})}{dt}\right)_{t=0} = \Gamma(\mathbf{z}-\mathbf{z}^*)$$

In other words, in the case of $\bar{\phi}_t(\mathbf{z})$ the formula (10) is not just an approximation, since it yields exactly the value of the derivative in question. For this reason the flow $\phi_t(\mathbf{z})$ can be approximated by the flow $\bar{\phi}_t(\mathbf{z})$ when **z** is close to $\mathbf{z}^*$.

The flow (12) is called *linear* because its field $\bar{Z}$ is linear in the sense that (13) is valid. The behavior of a linear flow around its fixed point can be characterized in a relatively simple manner in terms of its eigenvectors and eigenvalues. Clearly, if $\mathbf{z}_0 - \mathbf{z}^* = \mathbf{u}$ is an eigenvector of the matrix Γ which corresponds to the linear flow $\bar{\phi}_t$, the result (13)implies that as *t* grows, the change in $\bar{\phi}_t(\mathbf{z}_0)$ will have the direction of the vector **u**. In other words, the system moves towards or away from the steady state $\mathbf{z}^*$ along a straight line.

In general, the fixed point $\mathbf{z}^*$ is called *hyperbolic* if none of the eigenvalues of the matrix Γ has zero real part. In what follows, we shall restrict attention to hyperbolic flows. It is clear from (13) that the eigenvectors which correspond to the eigenvalues with a negative real part generate the *stable subspace* whose points **z** satisfy the condition

$$\lim_{t\to\infty}\bar{\phi}_t(\mathbf{z})=\mathbf{z}^*$$

(Arrowsmith & Place, 1990, pp. 64-70). We denote the stable subspace by $\bar{W}_s$ and we denote its number of dimensions by n_s. The case in which the eigenvalues corresponding to $\bar{W}_s$ had imaginary components seems irrelevant for economic applications, and from now on we only consider the case in which such eigenvalues are negative (rather than complex numbers with a negative real part).[3]

The *invariant manifold theorem for flows* (Arrowsmith & Place, 1990, p. 70 and p. 68) makes it possible to generalize these results to arbitrary flows. According to the invariant manifold theorem

$$W_s=\left\{\mathbf{z}\in\mathbb{R}^n\,\middle|\,\lim_{t\to\infty}\phi_t(\mathbf{z})=\mathbf{z}^*\right\}$$

is in a suitably chosen environment of $\mathbf{z}^*$ a manifold (which can be called the *stable manifold*) whose number of dimensions is n_s (i.e. the number of dimensions of the stable subspace $\bar{\phi}_t(\mathbf{z})$ of the corresponding linear flow) and which is tangent to $\bar{W}_s$ at the point $\mathbf{z}^*$.

These rather technical definitions and results can be illustrated with simplified version of the Ramsey model shown in Figures 1(a) and (b). In this case the flow $\phi_t(\mathbf{z})=\phi_t\left([k,\tilde{c}]^T\right)$ is determined by the time derivative $Z(\mathbf{z})=Z\left([k,\tilde{c}]^T\right)$ which is given by (6). The fixed point $\mathbf{z}^*=[k^*\ \tilde{c}^*]^T$ is the point for which $Z(\mathbf{z}^*)=0$ and as it was observed above, it is determined by (7). The Jacobian matrix is seen to be

$$\Gamma=\begin{bmatrix}\partial Z_k/\partial k & \partial Z_k/\partial c\\ \partial Z_c/\partial k & \partial Z_c/\partial c\end{bmatrix}_{k=k^*,c=c^*}=\begin{bmatrix}\rho & -1\\ f''(k^*)c^* & 0\end{bmatrix}$$

The eigenvalues of Γ are given by

[3] Analogously, the eigenvectors which correspond to the eigenvalues with a positive real part generate the *instable subspace*, in which $\bar{\phi}_t(\mathbf{z})$ grows without bounds as $t\to\infty$ but $\lim_{t\to-\infty}\bar{\phi}_t(\mathbf{z})=\mathbf{z}^*$.

$$\begin{cases} \lambda_1 = \rho/2 - \frac{1}{2}\sqrt{\rho^2 - 4f''(k^*)c^*} \\ \lambda_2 = \rho/2 + \frac{1}{2}\sqrt{\rho^2 - 4f''(k^*)c^*} \end{cases}$$

It is immediately observed that λ_2 is always positive, and that $\lambda_1 < 0$ if and only if $f''(k^*) < 0$, i.e. whenever capital has diminishing returns. In this case the model has precisely one eigenvalue which is smaller than 0, and this eigenvalue corresponds to a one-dimensional eigenspace. This eigenvector has the direction of the line L_1 in Figure 1(a). The line L_1 is also the stable set of the corresponding linearized flow $\bar{Z}$. As one may expect on the basis of the approximation (10), it is located quite close to L_1. In this context the invariant manifold theorem states simply that the stable set *S* has to be one-dimensional since L_1 is one-dimensional, and that S is tangent to L_1 at $\mathbf{z} = \mathbf{z}^*$.

On the other hand, both eigenvalues are positive in the case of Figure 1(b) in which $f''(k^*) > 0$. This implies that the stable set of linearized flow is zero-dimensional and consists of the single point $\mathbf{z} = \mathbf{z}^*$. In this case the invariant manifold theorem states that also the stable set of the flow (6) has the dimensionality zero implying that the system cannot reach the steady state if it is not in it originally.

As these results make clear, the uniqueness of the transition path is a rather specific feature of the neoclassical growth model: in general there is no reason why the stable set *S* could not be the whole plane (in which case the system would approach the steady state, however one chose the consumption $\tilde{c}$ for each value of the capital *k*), or consist only of the point z^*, as in Figure 1(b).

The invariant manifolds theorem is quite essential for result for our main result, which can be explained intuitively as follows. We consider the problem of finding a point $\mathbf{z} = [\mathbf{x}_0 \ \ \mathbf{y}]^T$ where $\mathbf{x}_0 = (x_{10}, \dots, x_{n_x 0})$ are the given values of the predetermined variables, which is located in the stable set of the flow *Z* that represents the model. To fix the vector **z**, one has to determine its $n = n_x + n_y$ components. If the number of dimensions of the stable manifold is n_s, the condition that a point is located in it can be formulated in terms of $(n - n_s)$ equations in a suitably chosen coordinate system. The statement that the predetermined variables have their given initial values can be formulated by n_x equations, implying that the point we are looking for is characterized by

$n-n_s+n_x$ equations. Hence, there will be the same number (*n*) of unknowns as there are equations if and only if $n-n_s+n_x=n$, i.e. if

(14) $$n_x=n_s$$

Hence, the transition path should be expected to be unique if (14) is valid, but not otherwise.

This argument can be given a rigorous formulation by considering first the linearized version of model, which is given by the linear flow $\bar{Z}$. In this case also the $(n-n_s)$ equations which state that the point $\mathbf{z}=[\mathbf{x}_0 \ \ \mathbf{y}]^T$ is in the stable set are linear, so that the choice of the *n* components of **z** is restricted by altogether $m=(n-n_s)+n_x$ linear equations. Clearly, when $n_s>n_x$, it must be the case that $m<n$, and if the *m* linear equations are not contradictory, they have an infinite number of solutions. If $n_s=n_x$ so that $m=n$ one can formulate required *n* conditions as a matrix equation of the form

(15) $$\Phi\mathbf{z}=\mathbf{b}$$

and observe that the solution exists whenever $\det\Phi\neq 0$. Finally, if $n_s<n_x$, there are more equations than unknowns and a solution can exist only if some of the equations are redundant. Finally, we note that in sufficiently small neighborhood of the steady state $\mathbf{z}^*$ these results must carry over (non-linear) flow *Z s*ince its stable set is tangent to the stable set of $\bar{Z}$ and has the same number of dimensions with it. This means that *generically,*[4] *the transition path exists and is unique in sufficiently small neighborhoods of the steady state when the number of predetermined variables* n_x *is identical with the number* n_s *of negative eigenvalues of the corresponding linearized flow* (13) *, but not otherwise*.

4. Varieties of endogenous growth

The origins of the endogenous, i.e. innovation-based growth models have been surveyed in (Aghion & Howitt, 1998) and (Aghion & Howitt, 2009). More recently, (Akcigit, 2023) has presented a short survey of the development of IBEG models, focusing on a few models which yield major new policy

[4] The word "generically" should be understood to as referring to the case in which (15) is valid, i.e. in which the matrix Φ has a non-zero determinant.

implications. We present our classification of endogenous growth models in Figure 2. While our choice of older models (until (Klette & Kortum, 2004)) is standard, the newer models have been picked up from a large literature with the aim of illustrating the various forms of the transition path problem in mind. Our classification includes the models that are considered in (Akcigit, 2023). Similarly with (Aghion & Howitt, 2009), our classification distinguishes between four growth paradigms: the neoclassical growth model, the AK model, the product-variety model, and the Schumpeterian model.

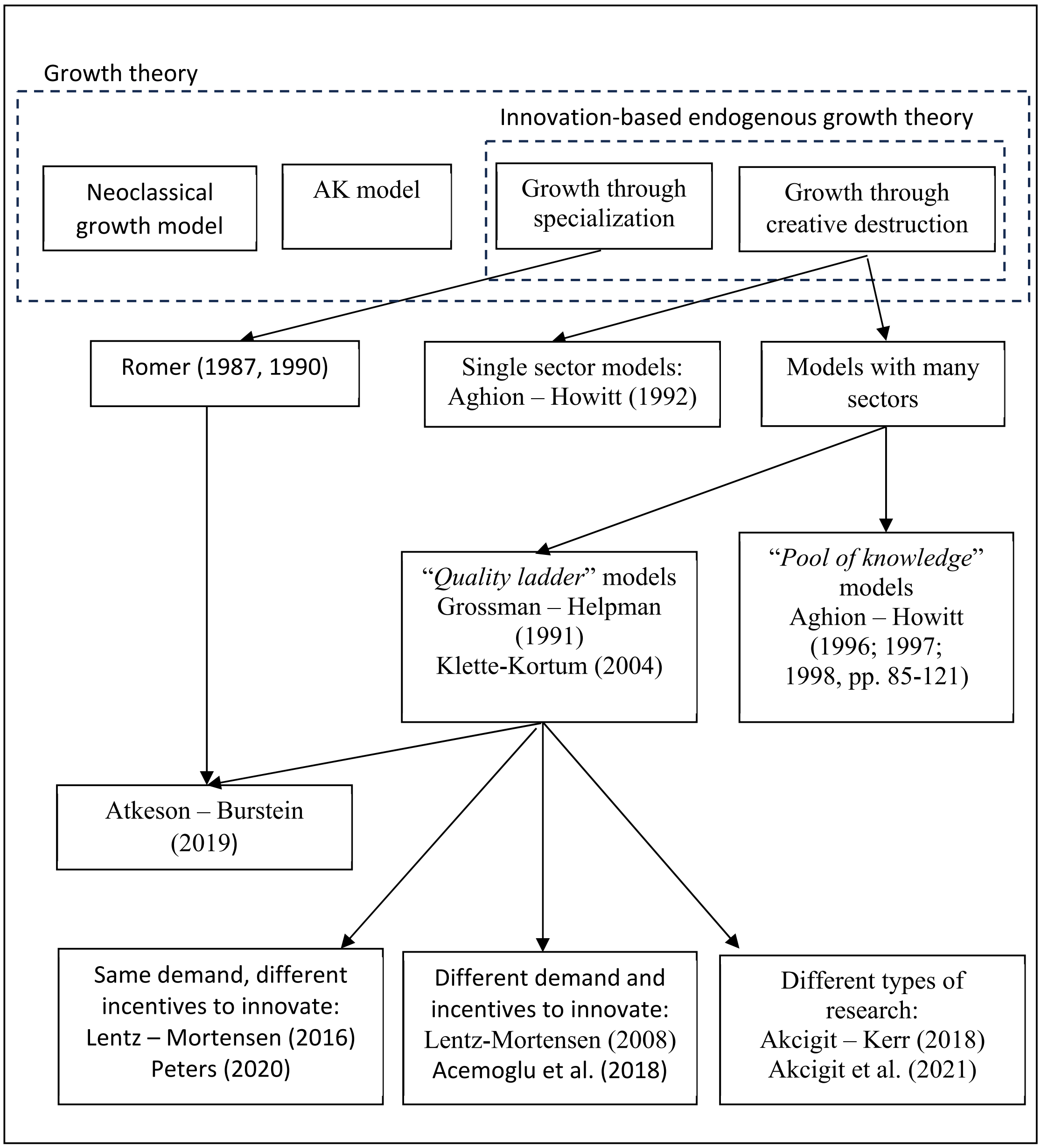

Figure 2. Development of innovation-based endogenous growth models

The neoclassical growth model views the growth of total factor productivity *A* as a wholly exogenous phenomenon. The *AK* model can be viewed as a first attempt to endogenize its growth, because in the *AK* model the growth of the total factor productivity *A* is an increasing function of the total amount of capital in the economy. This dependence has been meant to represent *learning by doing*. An early version of this model was put forward in (Frankel, 1962), and the much more general model in (Romer, 1986) included such dependence as its special case. In a Frankel type model firms do not invest in research or development, since technological progress is in them an unintended consequence of accumulation of capital on the aggregate level.

In innovation-based endogenous growth (IBEG) models the economy is divided into a production sector and a research sector, and it is assumed that the improvements in technology depend on the resources that the research sector is given. The relevant resources consist usually of specialized research labor. In these models the funding of the research sector is motivated by the monopoly rents from the new or improved goods that the researchers invent.

As Figure 2 illustrates, a major distinction can be drawn between the models in which the research and development causes *specialization*, i.e. the emergence of new products, and in which it causes *creative destruction*, i.e. emergence of higher quality versions of existing products which replace their old, lower quality versions. The first models of growth through specialization were put forward in (Romer, 1987, 1990). We shall illustrate our framework by applying it to the simplified version of the Romer framework which has been presented in (Aghion & Howitt, 1998).

The simplest possible model of growth through creative destruction is a model in which there is just a single sector and in which growth proceeds in an abrupt way whenever an innovation is made. A model of this type has been analyzed in detail in (Aghion & Howitt, 1992). The newer models of growth through creative destruction have usually a fixed but infinite number of sectors.

One may draw a further distinction between the *quality ladder models* in which an innovation in a sector increases the level of its technology (i.e. its level of productivity or its product quality) with a fixed step and the *pool of knowledge models* in which an innovation in a sector moves the sector to the technological frontier of the whole economy. The latter assumption can be interpreted as representing knowledge spillovers between sectors, and it has been elaborated in detail in (Aghion & Howitt, 1996; Aghion & Howitt, 1997; Aghion & Howitt, 1998).

However, pool knowledge models have gained little popularity in the subsequent literature. It is more customary to assume that there is a quality ladder, i.e. to assume that each innovation increases the quality of a good, or the productivity of the sector producing it, by a fixed amount. An early model of this kind was put forward in (Grossman & Helpman, 1991). In their setting each firm has just one product, and due to the choice of consumers' utility function, all firms face the same demand. As we shall shortly see, Grossman and Helpman's model has no predetermined state variables, implying that the economy could immediately "jump" to its balanced growth path if it were not in it originally. In addition, (Grossman & Helpman, 1991) prove that the model does not have any other equilibria besides the one which corresponds to the balanced growth path.

(Klette & Kortum, 2004) generalized Grossman and Helpman's setting into a model in which each firm has several products, and in which the firms can differ with respect to both their profitability and their innovative capacity. Despite of these complications, the dynamics of the model outside its balanced growth path remain analyzable because of the model's assumptions concerning research costs and innovation quality. However, the transition paths of the newer quality ladder models in Figure 2 pose more challenging problems.

Many of newer models in Figure 2 combine the Schumpeterian quality ladder approach with growth through specialization. This means that a new innovation can result in either the emergence of a new type of a product in a new sector, or in the replacement of an existing product with an improved product[5] in the same sector. This is the case in e.g. the model of (Atkeson & Burstein, 2019) that was discussed in Section 1.

Also the other newer papers which are mentioned in Figure 2 have a richer structure than the model of (Klette & Kortum, 2004). A richer model structure makes it possible to associate model parameters with a larger number of observable features of actual economies, but it also essentially limits the possibilities of considering transition paths which lead to long-run equilibria. As we shall see, in the case of some newer models the theory which we have developed could be used for analyzing transition paths, but for some models the situation is more complicated, because there is no obvious way to summarize the state of the economy by a finite number of state variables and control variables.

[5] More precisely, in some models the improvements are represented as quality improvements in the products of a sector, while in others they are represented as productivity improvements in their production.

It should be observed that there are important classes of endogenous growth models which do not naturally fall into any group of Figure 2. E.g. (Acemoglu et al., 2012) considers a model for the shift from dirty to clean technology. Under some plausible assumptions, the laissez-faire equilibrium is a situation in which all scientists work on dirty technology, leading to an environmental disaster, but this development can be avoided with a temporary subsidy to clean technology. In either case, the transition path of the model is trivial, as there is a fixed number of scientists, who all work within the field of either clean or dirty technology. Such models are outside the scope of this paper.

5. Models of growth through specialization

The growth models in (Romer, 1987) and (Romer, 1990) assumed that there is a single final good which is produced from an infinite variety of intermediate goods. Technological development consists of inventing new intermediate goods, and the investment into the research which is needed for inventing them is motivated by the monopoly profit that the inventor receives from them.

The following discussion of the model is based on the slightly simplified model version in (Aghion & Howitt, 1998, pp. 35-39). In the version of Aghion ja Howitt the production function is

(16) $$Y = L_W^{1-\alpha} \int_0^A x(i)^\alpha \, di$$

Here L_W is the labour used for producing the final good and *A* is the number of the available intermediate goods. The intermediate inputs $x(i)$ are produced using only capital, and $x(i)$ represents the amount of capital which is used for producing the good *i*. The final good is the numéraire. In equilibrium the same amount of capital is used for producing each intermediate input, and hence, $x(i) = x = K/A$, when *K* is the aggregate amount of capital. An increase in the number *A* of intermediate goods will increase output because

(17) $$Y = L_W^{1-\alpha} \int_0^A (K/A)^\alpha \, di = A^{1-\alpha} L_W^{1-\alpha} K^\alpha$$

The producer of each intermediate good participates in monopolistic competition with the producers of the other intermediate goods. Accordingly, the revenue of the producer of intermediate good *i* is

(18) $$R = \frac{d}{dx}\left[L_W^{1-\alpha} x^{\alpha}\right] x = \alpha L_W^{1-\alpha} x^{\alpha}$$

The interest rate is determined by the returns to capital and it is

(19) $$r = \frac{\partial R}{\partial x} = \alpha^2 L_W^{1-\alpha} x^{\alpha-1}$$

Economic growth is caused by both capital accumulation and the increase in the number of intermediate goods. When *C* is aggregate consumption, the amount of capital develops as

(20) $$\dot{K} = Y - C$$

Consumers are assumed to have an isoelastic utility function $u(c) = \left(c^{1-\eta} - 1\right) / (1-\eta)$, and hence, aggregate consumption will develop as

(21) $$\dot{C} = \frac{r-\rho}{\eta} C$$

where ρ is the rate of time preference.

The number of available intermediate products develops as

(22) $$\dot{A} = \delta L_R A$$

where L_R is the amount of research labor. In what follows we shall normalize the total amount of labor to 1, an hence,

(23) $$L_W + L_R = 1$$

The amount of labor in the two sector is determined by the condition that the wage *w* must be identical in production and in research.

Clearly, on a balanced growth path the aggregate capital *K* and the number of products *A* must grow at the same pace. A transition path corresponds in this case to a situation in which originally the ratio $x = K/A$ differs from its long-run value. (Romer, 1987, 1990) and (Aghion & Howitt, 1998) do not contain an analysis of this situation. We shall now apply the theory we presented in Section 3 above to it.

We shall use the ratio $x = K/A$ as a state variable, and we shall let the control variables be the normalized consumption $c = C/A$ and the amount of labor in production L_W. The time development of the ratio *x* is determined by (20) and (22), and the time development of normalized consumption c is determined by (21) and (22).

It is more challenging to represent the time development of L_W in terms of the variables *x*, *c* and L_W. The value *P* of patent to a new intermediate good must be the price of labour which is needed for producing it and hence,

$$P = w/(\delta A)$$

This will lead to an expression to the time derivative of the wage *w* when it is observed that

$$\dot{P} = rP - \pi$$

On the other hand, the wage must also be the marginal product of labor in production, which is determined by (16). Combining these observations, one arrives at the following equations which characterize the time development of the model:

(24)
$$\begin{bmatrix} \dot{x} \\ \dot{c} \\ \dot{L}_W \end{bmatrix} = Z(x, c, L_W) = \begin{bmatrix} L_W^{1-\alpha} x^{\alpha} - c - \delta x(1 - L_W) \\ \frac{1}{\eta}\left(\alpha^2 L_W^{1-\alpha} x^{\alpha-1} - \rho\right) - \delta c(1 - L_W) \\ L_W^{2-\alpha} x^{\alpha-1} - \frac{c}{x} L_w - \delta(1 - L_W) L_w - \delta L_W^2 \left[\frac{\alpha}{\delta} \frac{x^{\alpha-1}}{L_w^{\alpha}} - 1\right] \end{bmatrix}$$

In this formulation of the model the balanced growth path is represented by the fixed point of the mapping *Z*. This is the situation in which each of the derivatives in (24) is zero. Because there is just one predetermined state variable, the normalized amount of capital *x*, the uniqueness condition (14) now states that the negative eigenvalues of the Jacobian of (24) must span a one-dimensional vector space.

We have calculated the eigenvalues of the Jacobian of (24) for some plausibly chosen combinations of the parameters α, η, ρ and δ. It turned out that, when also the value of the shares of labor in production and research L_W and L_R are chosen plausibly, the Jacobian has two positive eigenvalues and a single negative eigenvalue.[6]

The significance of this result can be illustrated by contrasting the three-dimensional space of all combinations (x,c,L_W) to the two-dimensional space of capital-consumption combinations (k,c) of the neoclassical growh model of Figure 1. In the Ramsey model the transition path which leads to the fixed point (k^*,c^*) is a one-dimensional manifold. Accordingly, for each initial value of the capital *k* there is a single value of *c* for which the economy starts to approach its long-run equilibrium. In the Romer growth model the analogous space (x,c,L_W) is three-dimensional, but also in this space the manifold in which the economy approaches the fixed point is (at least for some plausible parameter combinations and the plausible balanced growth path corresponding to them) a one-dimensional curve. Hence, when the initial value of the ratio $x = K/A$ has been given, there is one and only one way to fix the initial values of the normalized consumption $c = C/A$ and the production labor L_W so that the system is on this curve. More precisely, this must be the case at least when the ratio *x* is sufficiently close to its balanced growth value.

6. A Schumpeterian endogenous growth model with a trivial transition path

In Figure 2 we divided the Schumpeterian growth models into pool of knowledge models and quality ladder models. As we saw above, pool of knowledge models have received little attention in the subsequent literature, and to the best of my knowledge, their dynamics have not been analyzed in detail.[7] In what follows we shall not consider them further.

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[6] We gave the parameter α values between 0.2 and 0.5; the parameter δ values between 0.01 and 0.3; the parameter ρ values between 0.01 ja 0.5; and the parameter η values between 1.05 and 1.5. It turned out that if the model is in an equilibrium in which labor is mostly employed in production rather than in research, the Jacobian which is calculated for the equilibrium values of *x*, *c* and L_W has two positive eigenvalues and a single negative eigenvalue. It also turned out that the model might also have another balanced growth path in which labor is mostly employed in research rather than in production and in which the ratio *x* is smaller. The Jacobian which corresponds to this implausible equilibrium has usually just a single positive eigenvalue.

[7] (Aghion & Howitt, 1998), which discusses pool of knowledge models at some length, does contain also a discussion of their dynamics, but this discussion is based on a constant savings rate (p. 110), and not on optimizing consumption over time.

While in a model of growth through specialization R&D results in designs for products of altogether new types, in a quality ladder model R&D creates designs for higher-quality versions of existing products or improved production methods that allow one to produce a larger quantity of an existing product with the same resources. These two formulations turn out to be equivalent. In what follows, our focus will be on the models with quality improvements.

The early Schumpeterian model in (Aghion & Howitt, 1992) was concerned with an economy with a single sector, but similarly with most of the subsequent quality ladder models, in the model of (Grossman & Helpman, 1991) the goods of the economy form an continuum $[0,1]$. The only factor of production is labor. All products have the same costs of production, as each unit of labor produces one unit of a product. In this case the production cost per unit equals the wage *w*.

Besides production, labor can be used for research, which creates new higher-quality designs for the goods. Each new design is invented by an entrepreneur, who receives a permanent monopoly for it. However, this monopoly becomes worthless when an even better design is invented. (Grossman & Helpman, 1991) assume that the utility function of the consumers is given by the equation

(25) $$U = \int_0^\infty e^{-\rho t} \ln u(t)\, dt$$

where the instantaneous utility function is given by

(26) $$\ln u(t) = \int_0^1 \ln\left(\sum\nolimits_k Q_k(j)\, x_k(j)\right) dj$$

Here $Q_k(j)$ is the quality of version *k* of the good *j*, and $x_k(j)$ is its output. Since all versions *k* have the same production costs, the entrepreneur with the highest-quality version of a good can force other producers out of the market. Hence, in equilibrium *u* is given by the following formula, which we shall dub the *demand assumption*:

(A-D) $$\ln u(t) = \int_0^1 \ln\left[Q(j)\, x(j)\right] dj$$

Here $Q(j)$ and $x(j)$ refer to the quality and output of the newest version of good *j*. Clearly, **(A-D)** implies that

(27) $$\ln u(t) = \int_0^1 \ln Q(j)\, dj + \int_0^1 \ln x(j)\, dj$$

It is easy to see that when the consumers maximize their utility, the wealth spent on each good is identical and hence,

(28) $$x(j)\,p(j) = const$$

where $p(j)$ is the price of the good *j*. In equilibrium the output $x(j)$ must be equal with the demand, and (27) shows that the equilibrium demand for good *j* is independent of its quality $Q(j)$

If the entrepreneur with the patent to the newest design of good *j* was not faced with competition, the good *j* would not have a well-defined equilibrium price, since according to (28) an increase in the price $p(j)$ would always increase the profit $\pi = x(j)\big(p(j) - w\big)$. However, the entrepreneur competes with the inventor of the previous design of the good. By assumption, each improvement to a product *j* improves its quality $Q(j)$ with the same multiplier *q*. The assumption (26) easily implies that if the inventor of the newest design for some good tried to sell it with a price higher than qw, the owner of the earlier design would prevent its sales by choosing a price which is sufficiently close to the wage *w*. Accordingly, the equilibrium price of each product is qw, and the price, the demand and the monopoly profit are identical for all products.

Just like in Romer's model the amount of labor in production is denoted by L_W and the amount of labor in research is denoted by L_R. The aggregate amount of labor is *L*, so that

(29) $$L_W + L_R = L$$

There is free entry to research, and by assumption, the hazard rate of creating an innovation is proportional to the amount of research labor. (Grossman & Helpman, 1991) consider the symmetric equilibrium in which there is the same amount of research in each sector. To unify notation with Section 7 below, I shall denote the amount of research labor which is needed for producing the hazard rate 1 of innovations by χ and the aggregate innovation rate of the economy by μ. Clearly, research labor is given by

(30) $$L_R = \chi\mu$$

Except for the quality distribution $Q(j)$, which does not affect the time development of the economy, all the variables of this model can be viewed as control variables whose values can immediately jump to their long-run balanced-growth values. Hence, the model always has an equilibrium. In addition, (Grossman & Helpman, 1991) demonstrate that the model does not have any other equilibria besides the balanced-growth equilibrium. Their argument is based on considering the equilibrium values of consumer expenditure *E* and a monopolist's risk per unit time of losing the monopoly, which in a symmetric equilibrium must be equal with μ.

Grossman and Helpman let wage be the numéraire. This implies that the consumption expenditure is given by

$$E = qL_W$$

Combining this result with (29) and (30), one can conclude that

(31) $$\frac{E}{q} + \chi\mu = L$$

In Figure 3 this resource constraint has been depicted as the straight line Λ in a (μ, E) coordinate system. On the other hand, when the utility function is the logarithmic function (25), utility maximization requires that

(32) $$\frac{\dot{E}}{E} = r - \rho$$

Here *r* is the interest rate, which must be determined by the value *v* of a patent (the only asset in the model). The expected income stream from a patent must on the one hand equal rv and on the other hand $\pi + \dot{v} - \mu v$ (profit from the corresponding design, change of the current value of patent, and expected loss from losing the patent). However, the value of a patent $v = w\chi = \chi$ must be constant. Hence,

(33) $$r = \frac{\pi}{v} - \mu = \frac{(1 - 1/q)E}{\chi} - \mu$$

Combining (32) and (33), one can conclude that

(34) $$\frac{\dot{E}}{E} = \frac{(1 - 1/q)E}{\chi} - \mu - \rho$$

Hence, the path on which the expenditure *E* stays constant must be a straight line Π in the coordinate system of Figure 3.

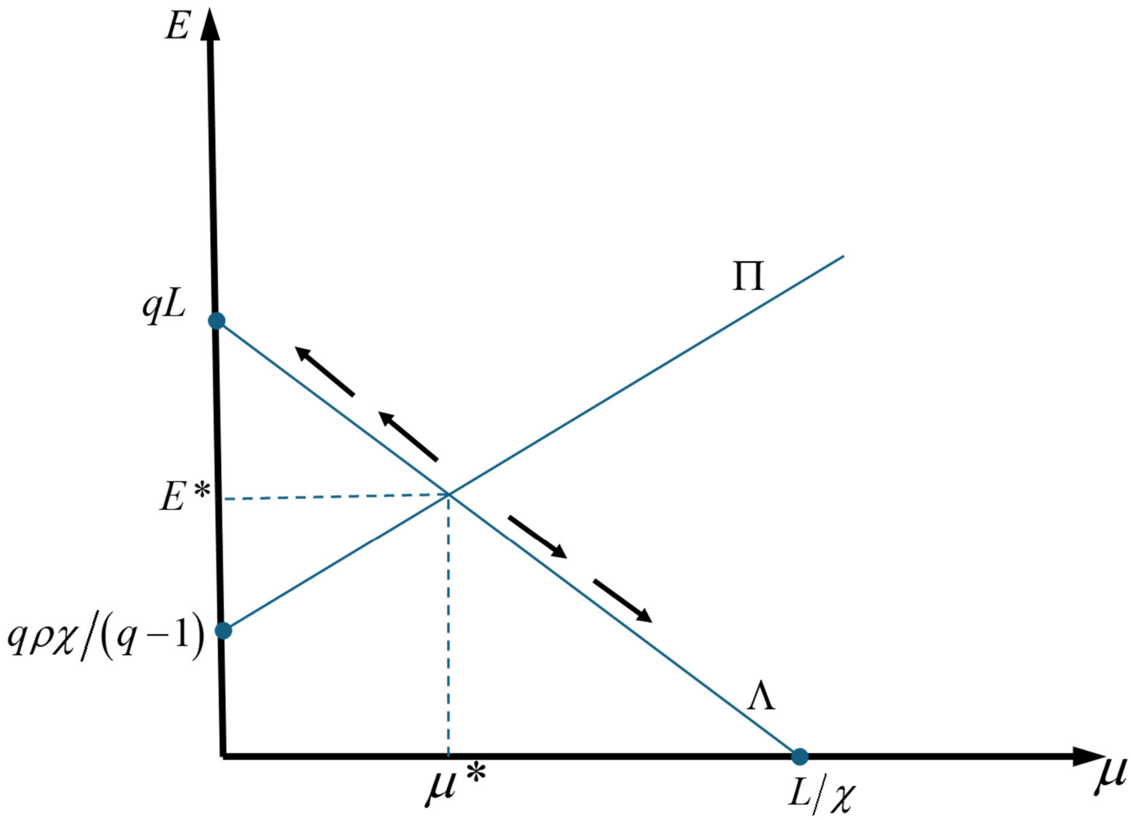

Figure 3. Equilibrium of the Grossman – Helpman quality ladder model

The two straight lines Λ and Π intersect at a single point in Figure 3, and this point of intersection (μ^*, E^*) represents the balanced growth path. It Is also immediately obvious from (34) that when the resource constraint (31) is valid but $E > E^*$, it must be the case that $\mu < \mu^*$ and that, according to (34), the derivative $\dot{E}$ is positive and the system moves away from its balanced growth equilibrium. The analogous conclusion is valid when $E < E^*$. Hence, one may conclude that the balanced growth equilibrium is the only equilibrium of the model. This makes the transition problem of the model trivial: if one added some structure which allowed for policy measures without changing the main assumptions of the model, after a policy change the model economy would simply immediately shift from one long-run equilibrium to another.

7. The generalization of Klette and Kortum

(Klette & Kortum, 2004) generalized the model of Grossman and Helpman into a setting in which each firm can have several products. When a firm's number of products is interpreted as a measure of its size, the generalized model can be used for analyzing the effects of innovation on market structures. (Klette & Kortum, 2004) consider several versions of their model. In some versions there are several types of firms which differ also with respect to the size of the innovations that they create and the "production function" which describes the costs of creating innovations In what follows I shall use ξ to denote the type of the firm. While the size of a firm can change in the Klette and Kortum model, the type ξ is assumed to be a constant which is characteristic for the firm. Below we shall see that solving Klette-Kortum model's transition path from a non-equilibrium market structure or non-equilibrium distribution of firm types would not pose very challenging problems.

Similarly with (Grossman & Helpman, 1991), Klette and Kortum's model assumes that the goods in the market form a continuum which stays constant and that technological progress takes the form of emergence of new designs for existing goods. Also the utility function is still given by (25), and the demand assumption (A-D)is valid. In what follows, the wage *w* will still be used as the numéraire. However, the innovative step q_ξ can now be different for different types ξ.

Firms are divided into incumbents and entrants. An entrant needs *F* researchers to create new designs at Poisson rate 1. Since the wage is the numéraire, this is also the cost of producing innovations at Poisson rate 1 for the entrant An entrant has a single product in its portfolio after entry. The entrant does not know the type ξ that it receives if it manages to enter (and which also determines the quality improvement q_ξ in its first product). However, the probability distribution of types ξ stays constant and is known to the entrant.

Each incumbent can practice both research and production of the designs it has invented. Through research the incumbent may create a new design for a product, allowing it to add the product to its portfolio. The size of the improvement $q = q_\xi$ is determined by firm type ξ. An incumbent can have several (any finite number *n*) products in its portfolio.

The labor force *L* satisfies

(35) $$L_W + L_R + L_S = L$$

where L_W is the labor force in production, L_R is the labor force employed in research by the incumbents, and L_S is the labor force employed in research by the entrants which are trying to launch new firms.

While in the model of Grossman and Helpman the emergence rate of innovations was simply proportional to research effort, (Klette & Kortum, 2004) assume that the incumbents' research efforts have diminishing returns. A firm of size *n* and type ξ needs $n\chi_\xi\left(I/n\right)$ researchers to create innovations at rate *I*. Since the wage is the numéraire, in this case also its research costs are given by

(A-R) $$G_\xi\left(I,n\right) = n\chi_\xi\left(I/n\right)$$

where $\chi'_\xi\left(\lambda\right) > 0$ and $\chi''_\xi\left(\lambda\right) < 0$. The number $\lambda = I/n$ measures innovation efforts per product, and it can be called the innovation intensity of the firm. We shall call **(A-R)** the research cost assumption, and the motives for making it will be become obvious below.

Assuming that the interest rate is *r*, the value of a firm of type ξ and size *n* is $V_\xi\left(n\right)$, that its profit from each of its products is π_ξ, and the probability of losing the monopoly to a product is μ, the value of the firm is seen to develop in accordance with the equation[8]

(36) $$rV_\xi\left(n\right) = \pi_\xi n - n\chi_\xi\left(\lambda\right) + n\lambda\left(V_\xi\left(n+1\right) - V_\xi\left(n\right)\right) - \mu n\left[V_\xi\left(n\right) - V_\xi\left(n-1\right)\right] + \frac{dV_\xi\left(n\right)}{dt}$$

where the research intensity λ has been chosen so that it maximizes the right-hand side. This result clarifies the motivation for making the research cost assumption **(A-R)**. This assumption implies that

[8] This corresponds to formula (3) in Klette and Kortum, 2004 (p. 993) except for the added last term. Klette and Kortum do not explicitly consider transition paths on which the value of an innovation might change; rather, they first consider partial equilibria and then general equilibria in which $V_\xi\left(n\right)$ is a constant. As I shall argue, this is legitimate in the model versions that Klette and Kortum (2004) consider, since in them $V_\xi\left(n\right)$ must be constant also when the state of the economy changes on the aggregate level. However, this would not be acceptable in some of the generalized versions that have been put forward in the subsequent literature. E.g., $V_\xi\left(n\right)$ would not necessarily be constant after a policy measure which has different effects on the innovation incentives of different types of firms.

the profit of the firm, the innovation costs, the expected added value from new product lines per unit time, and the expected loss from lost product lines per unit time are all proportional to firm size *n*. This allows us to conclude that the model has a solution in which the value of a firm is proportional to its number of products, and also that the optimal value of λ is independent of firm size.

To see why this is the case, we postpone the detailed discussion of μ and assume that it is constant. We note that if also the interest rate *r* and firm value stay constant, equation (36) has a solution of the form

$$V_n(\xi) = nv_\xi$$

where

(37) $$v_\xi = \frac{\pi_\xi - \chi_\xi(\lambda)}{r - \lambda + \mu}$$

(Klette & Kortum, 2004) assume that the firm types differ with respect to both the innovation size (which we denote by q_ξ) and the research cost function χ_ξ. Clearly, for given q_ξ the profit of the firm is given by

$$\pi_\xi = (q_\xi - 1) L_W$$

We denote the average value ofthe profit π_ξ (relative to the fixed ξ distribution of the firms) by $\overline{\pi}$ and the average value of $\chi_\xi(\lambda)$ by $\overline{\chi}(\lambda)$ for each λ. The function χ_ξ is by the assumption given by

(A-Q) $$\chi_\xi(\lambda) = \frac{\pi_\xi}{\overline{\pi}} \overline{\chi}(\lambda)$$

Intuitively, small innovations, which yield less profit, require less effort than large ones for the incumbent firms. We shall call the assumption **(A-Q)** the quality assumption, because its validity is based on quality differences between innovations and the corresponding differences in the markups in the different types of firms. The assumption **(A-Q)** makes the balanced growth path easier to solve, and it makes the transition paths trivial.

Until now, ξ has been just a label for a firm type, which could have been arbitrarily chosen, but now specify the representation of firm type so that

$$\xi = \frac{\pi_\xi}{\overline{\pi}} = \frac{q_\xi - 1}{\overline{q} - 1}$$

where $\overline{q}$ is the average innovative step. Remembering that on a balanced growth path $r = \rho$, the value of the firm (37) now becomes

(38) $$v_\xi = \xi \overline{v} = \xi \frac{\overline{\pi} - \overline{\chi}(\lambda)}{\rho - \lambda + \mu}$$

We also note that since the firm chooses λ in order to maximize v_ξ, the value of λ must be independent of also ξ. In other words, firms make the same investment into the research independently of their types.

The solution of the balanced growth paths of Klette and Kortum's model can now be completed as follows. First, we recapitulate that the average cost of obtaining a monopoly to an improved product is *F*, while the expected value of the monopoly is $\overline{v}$. Hence, in equilibrium $\overline{v} = F$. It is easy to see that (38) is maximized when

$$\overline{v} = \overline{\chi}'(\lambda)$$

which allows us to solve for λ . Since also number of research workers in each product line is $\overline{\chi}(\lambda)$, it must also equal the aggregate research labor L_R of the incumbents and we may solve L_R as $L_R = \overline{\chi}(\lambda)$.

On the other hand, given that the demands for the goods are identical, the average profit must be $\overline{\pi} = (\overline{q} - 1) L_w$, where L_W is the labor force In production. The hazard rate of an innovation by the entrants is FL_S , and hence, $\mu = \lambda + FL_S$. We are left with two unknowns L_S and L_W , and with the two equations (35) and

$$F = \overline{v} = \frac{\overline{\pi} - \overline{\chi}(\lambda)}{\rho - \lambda + \mu} = \frac{(\overline{q} - 1) L_W - L_R}{\rho + FL_S}$$

The balanced growth path of the model is determined by solving these two equations for L_S and L_W.

We note that the condition characterizing the balanced growth path would be valid even if the state of the economy differed from its long-run steady state with respect to the market structure. E.g., a group of *n* small firms with just a single product would make precisely the same pricing decisions and decisions concerning research investments as a large firm with *n* products would make. Accordingly, (Klette & Kortum, 2004) use their model also for discussing the process through which the market structure approaches its long-run steady state, when it is not in that state originally.

In particular, the equilibrium conditions would remain valid even if the market structure differed from the steady-state firm size distribution or the distribution of ξ values differed from the steady-state firm type distribution because of some earlier policy interventions. Since the other relevant features of the economy are characterized by control variables, their values could “jump” to the long-run values that they have under current policies.

7. Some models with non-trivial transition paths

The easiest way to illustrate the transition problem is, perhaps, to consider the model in (Lentz & Mortensen, 2016), since this model shares the research cost assumption **(A-R)** and the demand assumption **(A-D)**, but not the quality assumption **(A-Q)**.

The model of (Lentz & Mortensen, 2016) assumes that the utility function is given by (25) ja **(A-D)** just like (Grossman & Helpman, 1991). The research cost function satisfies the condition **(A-R)**, implying that the size distribution of the firms on the market is irrelevant to the future development of the economy. The firms are now divided into type-0 and type-1 firms, and the innovations made by type-1 firms are larger. However, the assumption **(A-Q)**, which states that smaller innovations have smaller research costs, is not valid. Rather, it is assumed that the research cost function (which in our notation would be denoted by $\chi(\lambda)$) is identical for both types of firms, implying that type-0 firms innovate less than type-1 firms.

There are still entrants and incumbents, just like in (Klette & Kortum, 2004), and fixed shares of them are assigned types 0 and 1 after entry. In addition, the firms can change their types at an exogenously given rate. Hence, each high-type firm can have low-type products (i.e. products with a low innovative step and markup) in its portfolio, and vice versa. (Lentz & Mortensen, 2016) let K_{ij} $(i, j = 0,1)$ represent the number of type *i* products that are produced by firms which are currently of type *j*.

While the market structure (i.e. the number of products of each firm) does not affect the economy's future development, the aggregate number of products of low-type and high-type firms does affect it. Given that the numbers K_{ij} sum up to one, one can choose any three of them to be the predetermined variables of the considered economy. The question whether the values of K_{ij} approach their long-run equilibrium values also when they do not have these values originally is non-trivial, and it is not answered by (Lentz & Mortensen, 2016), as it has earlier been pointed out by (Güner, 2023).

Rather, Lentz and Mortensen solve only the long-run equilibrium of the model and contrast it with the development chosen by a social planner. This approach does not answer the more interesting question how the economy would react to policy reforms that affect but do not replace market economy. If one considered e.g. a hypothetical reform in which the share ϕ_0 of low-type firms created by entrants would be lowered (say, through educational reforms or by support offered to high-type entrepreneurs only), the values K_{ij} would no longer correspond to the long-run equilibrium relative to ϕ_0. To analyze this more interesting case, a discussion of transition paths would be needed.

An analogous point applies to (Peters, 2020). This model shares the assumption **(A-D)**, which would yield the same demand for all inputs independently of their quality, if they had the same markup. However, in Peter's setting a firm can make several subsequent quality improvements to the same product, implying that the markups can be different. (Peters, 2020) views the markup dispersion as a welfare-relevant measure of misallocation, and the model can be used for analyzing the welfare loss defined in terms of misallocation in various stationary equilibria. However, it is not easy to see how the transition would proceed in the model if some policy measure changed the research incentives of firms in a way that (say) in the stationary equilibrium markup heterogeneity would be smaller.

The earlier model in (Lentz & Mortensen, 2008) serves to illustrate the fundamentally different difficulties which emerge when (A-D) is not valid and the demand for different quality products can differ. While we have until now thought of the quality differences between products as differences in the utility that they produce, Lentz and Mortensen assume that there are intermediate inputs indexed by *i,* whose qualities at time t are denoted by $A_i(t)$. The intermediate inputs are used for producing a consumption good in accordance with the CES production function

$$C_t = \left[\int_0^1 \alpha(j) \left(A_t(j) x_t(j) \right)^{\sigma/(\sigma-1)} dj \right]^{(\sigma-1)/\sigma} \tag{39}$$

where $\alpha(j)$ is a time-independent distribution and $x_t(j)$ is the demand for input *j*. Clearly, now the profit of the producer of input *j* will depend on the quality of the other inputs.

In this model there are firms of different types, which differ with respect to the size of the innovations that they create. The research costs of the firm are still given by (A-R), but now these costs are independent of firm type. This implies that both the types of incumbents and the quality distribution of the available products will essentially affect each firm's incentives to innovate.

(Lentz & Mortensen, 2008) solve the long-run equilibrium of their model, but they do not solve its transition paths. Yet is clear that many interesting policy questions could be answered only by discussing them. E.g., one might consider a situation in which the markups would on the average be smaller than they are in the long-run equilibrium due to some earlier policy which had favored low-type firms. Now the problem of solving the transition path would be essentially more difficult than in the case of (Lentz & Mortensen, 2016) model (to which, as we saw, the theory of dynamical systems could be readily applied), since the transition path would depend on the whole distribution $A_i(t)$ rather than on a finite number of predetermined variables.

Also the remaining models that are mentioned in Figure 2 contain production functions which are incompatible with the demand assumption (A-D), implying that the relationship between the demand for a good and the qualities of the other goods is quite complicated in these models. (Acemoglu et al., 2018) put forward a model in which the firms differ with respect to their innovative capacities and in which policies favoring high-type firms (i.e. the firms with a larger innovative capacity) can lead to considerable welfare gains. (Akcigit & Kerr, 2018) draw a distinction between *exploitative* R&D, with which a firm tries to improve the quality of its products, and *explorative* R&D

which might lead to larger innovations and to acquiring new product lines. (Akcigit et al., 2021) introduces a distinction between applied and fundamental research into a somewhat more complicated setting. Both applied and fundamental research yield innovations which improve a single production line in a single industry, but a fundamental innovation will increase the sizes of the innovations that will subsequently be made to the other production lines of the same industry, and it might also create innovations in other industries.

Given the results of (Atkeson & Burstein, 2019), it is puzzling how little attention has been paid to the transition paths of these models in the relevant research. For example, (Acemoglu et al., 2018) provides quantitative estimates for the welfare effects of various policy measures (such as e.g. subsidizing R&D by incumbents or subsidizing firm entry) in the context of their model. However, the authors do not analyze the transition paths which would be associated with the introduction of such policies. This is made even more puzzling by the fact that the suggested policies would not change the long-run equilibrium distribution of relative product qualities, to the analysis of which the tools of dynamic systems theory would be badly suited.[9] Rather, the out-of-long-run-equilibrium features of the economy after the considered reforms could be analyzed in terms of a finite set of predetermined variables, such as the share of low-type firms among incumbents. However, there has been to the best of our knowledge no such analysis in the relevant literature.

9. Conclusion

We have reviewed the development of innovation-based endogenous growth models (IBEG models) and considered the transition paths which lead from one long-run equilibrium to another one in them. The economists who formulated the IBEG first models admitted the importance of determining their transition paths. While presenting the balanced growth path equilibrium of his specialization model, Paul M. Romer pointed out that his analysis of the growth though specialization model neglects the “transient dynamics” which emerge when the ratio K/A does not have its long-run equilibrium value. He stated that “one should be able to study convergence to the

[9] In the model of Acemoglu et al. (2018) the quality distribution of the products affects their consumption in a complicated way, since the consumers are maximizing a consumption aggregate (2) in ibid. However, since the innovations of high-type and low-type firms are of the same size in the model (see (10) in ibid.), a reform which favors high-type firms would not change the long-run equilibrium distribution of product quality.

balanced growth ratio of K to A using the tools used for studying the Solow and Uzawa models" (Romer, 1990, p. S90), although he had not attempted such analysis himself. As we saw above, the early Grossman-Helpman quality ladder model had a trivial transition path, since none of its variables are predetermined and since they can immediately "jump" to their equilibrium values, but (Grossman & Helpman, 1991) nevertheless presented a detailed argument which proved that no other equilibria exist.

However, no similar analyses are to the best of our knowledge available in the context of many important newer IBEG models. Above we saw that the theory of dynamical systems provides mathematical tools with which one can analyze the transition paths of many IBEG models, and we presented an analysis of this kind for Romer's model of growth through specialization. Turning to growth through creative destruction, we noted that the dynamics of the quality ladder model of (Klette & Kortum, 2004) was easy to analyze, although its structure was essentially more complicated than that of the Grossman-Helpman model. This was due to three assumptions: the demand assumption (A-D), the research cost assumption (A-R), and the quality assumption (A-Q).

Only some but not all of these assumptions are valid in most of the newer IBEG models with quality ladders. This turns the analysis of their transition dynamics into a non-trivial problem. While in some cases, like in the case of e.g. (Lentz & Mortensen, 2016), the transition dynamics of a model could be formulated as a dynamic systems theory problem (i.e. as a problem of determining the time development of finite set of predetermined variables and control variables), in some others the problem is more complicated, since the predetermined features of the economy can only be specified as a distribution on the real axis.

The problem of determining transition paths has a practical and a foundational dimension. (Atkeson & Burstein, 2019) have pointed out, using a model which nests many important IBEG models as its special cases, that the policy measures which considerably increase welfare in the long run might have moderate impact within the kind of time periods (e.g. next 20 years) that R&D policy measures are normally expected to be useful. Hence, when analyzing the impact of R&D policy measures, one should consider not just their long-term effects but also the transition period during which they begin to influence the economy.

The more fundamental dimension of the problem of transition paths is connected with the motives for studying IBEG models. Because of their highly idealized features, the only sensible interpretation of IBEG models is an instrumentalist one. To follow the terminology of Robert Lucas, one can call them *imitation economies*. The study of an imitation economy instead of a real one can be motivated by presenting explicit rules, based on the aims and capabilities of the agents in it, for deducing its behavior in various circumstances and by claiming that in relevant respects the imitation economy resembles real ones (cf. Lucas, 1980). However, it is not quite clear what the motive for replacing an actual economy with an imitation economy could be if the imitation economy resembles real economies also in so far that its development is undeducible.

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